

EXISTENCE OF A CLASS OF ROTOPULSATORS

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ABSTRACT. We prove the existence of a class of rotopulsators for the n -body problem in spaces of constant curvature of dimension $k \geq 2$.

1. INTRODUCTION

By n -body problems, we mean problems where we want to find the dynamics of n point particles. If the space in which such a problem is defined is a space of zero curvature, then we call any solution to such a problem for which the point particles describe the vertices of a polytope that retains its shape over time (but not necessarily its size) a homographic orbit.

A rotopulsator, also known as a rotopulsating orbit, is a type of solution to an n -body problem for spaces of constant curvature $\kappa \neq 0$ that extends the definition of homographic orbits to spaces of constant curvature (see [7]).

Homographic orbits (and therefore rotopulsators) can be used to determine the geometry of the universe locally (see for example [4], [7]).

In this paper, we will prove the existence of a subclass of rotopulsators that form a natural generalisation of orbits found in [4] and [5].

While this paper mainly builds on results obtained in [4], [5] and [22], research on n -body problems for spaces of constant curvature goes back to Bolyai [1] and Lobachevsky [19], who independently proposed a curved 2-body problem in hyperbolic space \mathbb{H}^3 in the 1830s. In later years, n -body problems for spaces of constant curvature have been studied by mathematicians such as Dirichlet, Schering [20], [21], Killing [12], [13], [14] and Liebmann [16], [17], [18]. More recent results were obtained by Kozlov, Harin [15], but the study of n -body problems in spaces of constant curvature for the case that $n \geq 2$ started with [9], [10], [11] by Diacu, Pérez-Chavela, Santoprete. Further results for the $n \geq 2$ case were then obtained by Cariñena, Rañada, Santander [2], Diacu [3], [4], [5], Diacu, Kordlou [7], Diacu, Pérez-Chavela [8]. For a more detailed historical overview, please see [4], [5], [6], [7], or [9].

In this paper, we will prove the following two theorems:

Theorem 1.1. *For any rotopulsating solution of (2.2) formed by vectors $\{\mathbf{q}_i\}_{i=1}^n$ as defined in (2.3), the vectors $\{\mathbf{Q}_i\}_{i=1}^n$ have to form a regular polygon if ρ is non-constant.*

Theorem 1.2. *Rotopulsating orbits formed by vectors $\{\mathbf{q}_i\}_{i=1}^n$ as defined in (2.3) exist if the vectors $\{\mathbf{Q}_i\}_{i=1}^n$ form a regular polygon.*

To prove these theorems, we will use a method strongly inspired by [4], [5] and [22]. Specifically, we will first deduce a necessary and sufficient criterion for the existence of rotopulsators. This will be done in section 2. We will then prove Theorem 1.1 and Theorem 1.2 in section 3 and section 4 respectively.

2. A CRITERION FOR THE EXISTENCE OF ROTOPULSATORS

In this section, we will formulate a necessary and sufficient criterion for the existence of rotopulsating orbits of the type described in (2.3).

Consider the n -body problem in spaces of constant curvature $\kappa \neq 0$.

As has been shown in [6], we may assume that κ equals either -1 , or 1 .

We will denote the masses of its n point particles to be $m_1, m_2, \dots, m_n > 0$ and their positions by the k -dimensional vectors

$$\mathbf{q}_i^T = (q_{i1}, q_{i2}, \dots, q_{ik}) \in \mathbf{M}_\kappa^{k-1}, \quad i = \overline{1, n}$$

where

$$\mathbf{M}_\kappa^{k-1} = \{(x_1, x_2, \dots, x_k) \in \mathbb{R}^k \mid \kappa(x_1^2 + x_2^2 + \dots + x_{k-1}^2 + \sigma x_k^2) = 1\}, \quad k \in \mathbb{N}$$

and

$$\sigma = \begin{cases} 1 & \text{for } \kappa > 0 \\ -1 & \text{for } \kappa < 0 \end{cases}.$$

Furthermore, consider for m -dimensional vectors $\mathbf{a} = (a_1, a_2, \dots, a_m)$,

$\mathbf{b} = (b_1, b_2, \dots, b_m)$ the inner product

$$(2.1) \quad \mathbf{a} \odot_m \mathbf{b} = a_1 b_1 + a_2 b_2 + \dots + a_{m-1} b_{m-1} + \sigma a_m b_m.$$

Then, following [3], [4], [5], [9], [10], [11] and the assumption that $\kappa = \pm 1$ from [6], we define the equations of motion for the curved n -body problem as the dynamical system described by

$$(2.2) \quad \ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^n \frac{m_j [\mathbf{q}_j - (\sigma \mathbf{q}_i \odot_k \mathbf{q}_j) \mathbf{q}_i]}{[\sigma - (\mathbf{q}_i \odot_k \mathbf{q}_j)^2]^{\frac{3}{2}}} - (\sigma \dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad i = \overline{1, n}.$$

Let

$$T(t) = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix}$$

be a 2×2 rotation matrix, where $\theta(t)$ is some real valued, twice continuously differentiable, scalar function, for which $\theta(0) = 0$.

Furthermore, let $\rho(t)$ be a nonnegative, twice continuously differentiable, scalar function.

We will consider rotopulsating orbit solutions of (2.2) of the form

$$(2.3) \quad \mathbf{q}_i(t) = \begin{pmatrix} \rho(t) T(t) \mathbf{Q}_i \\ Z(t) \end{pmatrix}$$

where $\mathbf{Q}_i \in \mathbb{R}^2$ is a constant vector and $Z(t) \in \mathbb{R}^{k-2}$ is a twice differentiable, vector valued function.

Finally, before formulating our criterion, we need to introduce some notation and a lemma:

Let $m \in \mathbb{N}$. Let $\langle \cdot, \cdot \rangle_m$ be the Euclidean inner product on \mathbb{R}^m and let $\|\cdot\|_m$ be the Euclidean norm on \mathbb{R}^m . Let $i, j \in \{1, \dots, n\}$. By construction $\|\mathbf{Q}_i\|_2 = \|\mathbf{Q}_j\|_2$ for all $i, j \in \{1, \dots, n\}$ and we will assume that $\|\mathbf{Q}_i\|_2 = 1$. Let β_i be the angle between \mathbf{Q}_i and the first coordinate axis. The lemma we will need to prove our criterion is:

Lemma 2.1. *The functions ρ and θ , are related through the following formula: $\rho^2(t) \dot{\theta}(t) = \rho^2(0) \dot{\theta}(0)$.*

Proof. In [5], using the wedge product, Diacu proved that

$$\sum_{i=1}^n m_i \dot{\mathbf{q}}_i \wedge \mathbf{q}_i = \mathbf{c}$$

where \mathbf{c} is a constant bivector.

If $\{\mathbf{e}_i\}_{i=1}^k$ are the standard base vectors in \mathbb{R}^k , then we can write \mathbf{c} as

$$(2.4) \quad \mathbf{c} = \sum_{i=1}^k \sum_{j=1}^k c_{ij} \mathbf{e}_i \wedge \mathbf{e}_j.$$

where $\{c_{ij}\}_{i=1,j=1}^k$ are constants. As $\mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i$ and $\mathbf{e}_i \wedge \mathbf{e}_i = 0$ (see [5]), for $i, j \in \{1, \dots, n\}$, we can rewrite (2.4) as

$$(2.5) \quad \mathbf{c} = \sum_{i=1}^k \sum_{j=i+1}^k C_{ij} \mathbf{e}_i \wedge \mathbf{e}_j$$

where $C_{ij} = c_{ij} - c_{ji}$.

Calculating C_{12} , will give us our result:

Note that

$$(2.6) \quad T^T = T^{-1} \text{ and } \dot{T} = \dot{\theta} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T$$

and

$$(2.7) \quad \begin{aligned} C_{12} &= \sum_{i=1}^n m_i (q_{i1} \dot{q}_{i2} - q_{i2} \dot{q}_{i1}) \\ &= \sum_{i=1}^n m_i (q_{i1}, q_{i2}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_{i1} \\ \dot{q}_{i2} \end{pmatrix}. \end{aligned}$$

Using (2.3) with (2.7) gives

$$(2.8) \quad \begin{aligned} C_{12} &= \sum_{i=1}^n m_i \rho^2 (Q_{i1}, Q_{i2}) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{T} \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} \\ &+ \sum_{i=1}^n m_i \rho \dot{\rho} (Q_{i1}, Q_{i2}) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} &\rho \dot{\rho} (Q_{i1}, Q_{i2}) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} \\ &= \frac{\dot{\rho}}{\rho} (q_{i1}, q_{i2}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q_{i1} \\ q_{i2} \end{pmatrix} = 0. \end{aligned}$$

So, using (2.6) repeatedly, we get that

$$\begin{aligned} C_{12} &= \sum_{i=1}^n m_i \rho^2 \dot{\theta} (Q_{i1}, Q_{i2}) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} + 0 \\ &= \sum_{i=1}^n m_i \rho^2 \dot{\theta} (Q_{i1}, Q_{i2}) T^T T \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} \\ &= \sum_{i=1}^n m_i \rho^2 \dot{\theta} (Q_{i1}, Q_{i2}) \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} \end{aligned}$$

which means that

$$(2.9) \quad C_{12} = \rho^2 \dot{\theta} \sum_{i=1}^n m_i (Q_{i1}^2 + Q_{i2}^2).$$

As, by construction

$$\sum_{i=1}^n m_i (Q_{i1}^2 + Q_{i2}^2) > 0,$$

we may divide both sides of (2.9) by

$$\sum_{i=1}^n m_i (Q_{i1}^2 + Q_{i2}^2),$$

which gives that

$$\rho^2 \dot{\theta} = \frac{C_{12}}{\sum_{i=1}^n m_i (Q_{i1}^2 + Q_{i2}^2)},$$

which is constant, so $\rho^2 \dot{\theta} = \rho^2(0) \dot{\theta}(0)$. \square

We now have the following necessary and sufficient criterion for the existence of a rototulsating orbit, as described in (2.3):

Criterion 1. Let

$$(2.10) \quad b_i = \sum_{j=1, j \neq i}^n \frac{m_j (1 - \cos(\beta_i - \beta_j))^{-\frac{1}{2}}}{(2 - \sigma \rho^2 (1 - \cos(\beta_i - \beta_j)))^{\frac{3}{2}}}.$$

Then necessary and sufficient conditions for the existence of a rototulsating orbit of non-constant size are that $b_1 = b_2 = \dots = b_n$ and

$$(2.11) \quad 0 = \sum_{j=1, j \neq i}^n \frac{m_j \sin(\beta_i - \beta_j)}{(1 - \cos(\beta_i - \beta_j))^{\frac{3}{2}} (2 - \sigma \rho^2 (1 - \cos(\beta_i - \beta_j)))^{\frac{3}{2}}}$$

for all $i \in \{1, \dots, n\}$.

Proof. Note that

$$(2.12) \quad \dot{T} = \dot{\theta} T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and consequently

$$(2.13) \quad \ddot{T} = \ddot{\theta} T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \dot{\theta}^2 T.$$

Inserting (2.3) into (2.2) and using (2.12) and (2.13) gives for the first and second lines of (2.2) that

$$(2.14) \quad \begin{aligned} & T \left(\ddot{\rho} I_2 + 2\dot{\rho}\dot{\theta} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \rho \left(\ddot{\theta} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \dot{\theta}^2 I_2 \right) \right) \mathbf{Q}_i \\ &= \rho T \left(\sum_{j=1, j \neq i}^n \frac{m_j [\mathbf{Q}_j - (\sigma \mathbf{q}_i \odot_k \mathbf{q}_j) \mathbf{Q}_i]}{[\sigma - (\mathbf{q}_i \odot_k \mathbf{q}_j)^2]^{\frac{3}{2}}} - (\sigma \dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i) \mathbf{Q}_i \right) \end{aligned}$$

where I_2 is the 2×2 identity matrix.

For the last $k-2$ lines, we get

$$(2.15) \quad \ddot{Z} = \left(\sum_{j=1, j \neq i}^n \frac{m_j [1 - (\sigma \mathbf{q}_i \odot_k \mathbf{q}_j)]}{[\sigma - (\mathbf{q}_i \odot_k \mathbf{q}_j)^2]^{\frac{3}{2}}} - (\sigma \dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i) \right) Z.$$

Note that

$$(2.16) \quad \mathbf{q}_i \odot_k \mathbf{q}_j = \rho^2 \langle \mathbf{Q}_i, \mathbf{Q}_j \rangle_2 + Z \odot_{k-2} Z.$$

As we have that $\langle \mathbf{Q}_i, \mathbf{Q}_i \rangle_2 = 1$ and as by (2.16),

$$\sigma^{-1} = \mathbf{q}_i \odot_k \mathbf{q}_i = \rho^2 \langle \mathbf{Q}_i, \mathbf{Q}_i \rangle_2 + Z \odot_{k-2} Z,$$

we may rewrite (2.16) as

$$\mathbf{q}_i \odot_k \mathbf{q}_j = \sigma^{-1} + \rho^2 \langle \mathbf{Q}_i, \mathbf{Q}_j \rangle_2 - \rho^2,$$

which can, in turn, be written as

$$(2.17) \quad \mathbf{q}_i \odot_k \mathbf{q}_j = \sigma^{-1} + \rho^2 (\cos(\beta_i - \beta_j) - 1).$$

Furthermore,

$$(2.18) \quad \dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i = \langle \dot{\rho} T \mathbf{Q}_i + \rho \dot{T} \mathbf{Q}_i, \dot{\rho} T \mathbf{Q}_i + \rho \dot{T} \mathbf{Q}_i \rangle_2 + \dot{Z} \odot_{k-2} \dot{Z}.$$

As T is a rotation in \mathbb{R}^2 , it is a unitary map, meaning that for $v, w \in \mathbb{R}^2$, $\langle Tv, Tw \rangle_2 = \langle v, w \rangle_2$, meaning that (2.18) can be written as

$$(2.19) \quad \dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i = \langle \dot{\rho} \mathbf{Q}_i + \rho T^{-1} \dot{T} \mathbf{Q}_i, \dot{\rho} \mathbf{Q}_i + \rho T^{-1} \dot{T} \mathbf{Q}_i \rangle_2 + \dot{Z} \odot_{k-2} \dot{Z}.$$

Using (2.12) with (2.19) gives

$$(2.20) \quad \begin{aligned} \dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i &= \dot{\rho}^2 + 2\rho\dot{\rho}\dot{\theta} \left\langle \mathbf{Q}_i, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{Q}_i \right\rangle_2 + \rho^2 \dot{\theta}^2 \|\mathbf{Q}_i\|^2 + \dot{Z} \odot_{k-2} \dot{Z} \\ &= \dot{\rho}^2 + 0 + \rho^2 \dot{\theta}^2 + \dot{Z} \odot_{k-2} \dot{Z}. \end{aligned}$$

Inserting (2.20) and (2.17) into (2.14) and multiplying both sides by T^{-1} provides us with

$$(2.21) \quad \begin{aligned} & \left((\ddot{\rho} - \rho \dot{\theta}^2) I_2 + (2\dot{\rho}\dot{\theta} + \rho \ddot{\theta}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \mathbf{Q}_i \\ &= \rho \sum_{j=1, j \neq i}^n \frac{m_j [\mathbf{Q}_j - (1 - \sigma \rho^2 (1 - \cos(\beta_i - \beta_j))) \mathbf{Q}_i]}{[\rho^2 (1 - \cos(\beta_i - \beta_j)) (2 - \sigma \rho^2 (1 - \cos(\beta_i - \beta_j)))]^{\frac{3}{2}}} \\ & \quad - (\sigma \rho \dot{\rho}^2 + \sigma \rho^3 \dot{\theta}^2 + \sigma \rho \dot{Z} \odot_{k-2} \dot{Z}) \mathbf{Q}_i. \end{aligned}$$

Taking the Euclidean inner product with \mathbf{Q}_i on both sides of (2.21) and using that $\|Q_i\|_2 = \|Q_j\|_2 = 1$, provides us with

$$(2.22) \quad \begin{aligned} & \ddot{\rho} - \rho\dot{\theta}^2 + \sigma\rho\dot{\rho}^2 + \sigma\rho^3\dot{\theta}^2 + \sigma\rho\dot{Z} \odot_{k-2} \dot{Z} \\ &= \left(\sigma - \frac{1}{\rho^2}\right) \sum_{j=1, j \neq i}^n \frac{m_j [(1 - \cos(\beta_i - \beta_j))^{-\frac{1}{2}}]}{[(2 - \sigma\rho^2(1 - \cos(\beta_i - \beta_j)))]^{\frac{3}{2}}}. \end{aligned}$$

Taking the Euclidean inner product of (2.21) with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Q}_i$ and using that $\|Q_i\|_2 = \|Q_j\|_2 = 1$ gives that

$$(2.23) \quad 2\rho\dot{\theta} + \rho\ddot{\theta} = \sum_{j=1, j \neq i}^n \frac{m_j \sin(\beta_i - \beta_j)}{[(1 - \cos(\beta_i - \beta_j))(2 - \sigma\rho^2(1 - \cos(\beta_i - \beta_j)))]^{\frac{3}{2}}}.$$

Let

$$\begin{aligned} b_i &:= \sum_{j=1, j \neq i}^n \frac{m_j [(1 - \cos(\beta_i - \beta_j))^{-\frac{1}{2}}]}{[(2 - \sigma\rho^2(1 - \cos(\beta_i - \beta_j)))]^{\frac{3}{2}}} \text{ and} \\ c_i &:= \sum_{j=1, j \neq i}^n \frac{-m_j \sin(\beta_i - \beta_j)}{[(1 - \cos(\beta_i - \beta_j))(2 - \sigma\rho^2(1 - \cos(\beta_i - \beta_j)))]^{\frac{3}{2}}}. \end{aligned}$$

Inserting (2.20) and (2.17) into (2.15), combined with (2.22) and (2.23) gives the following system of differential equations:

$$(2.24) \quad \begin{cases} \ddot{\rho} = \rho\dot{\theta}^2 - \sigma\rho\dot{\rho}^2 - \sigma\rho^3\dot{\theta}^2 - \sigma\rho\dot{Z} \odot_{k-2} \dot{Z} + \left(\sigma - \frac{1}{\rho^2}\right) b_i \\ \ddot{\theta} = \frac{c_i}{\rho} - 2\frac{\dot{\rho}}{\rho}\dot{\theta} \\ \ddot{Z} = \left(b_i - \sigma\dot{\rho}^2 - \sigma\rho^2\dot{\theta}^2 - \sigma\dot{Z} \odot_{k-2} \dot{Z}\right) Z \end{cases}$$

For (2.24) to make sense, we need that

$$(2.25) \quad b_1 = \dots = b_n \text{ and } c_1 = \dots = c_n$$

which shows the necessity of (2.25).

Furthermore, that (2.24) has a global solution holds by the same argument as the argument used in the proof of Criterion 1 in [4] to prove global existence of a solution of (15) and (17). By the uniqueness of solutions to ordinary differential equations given suitable initial conditions, the solution to (2.24) must be a rototranslating orbit, as every step from (2.14) and (2.15) to (2.24) is invertible.

Thus (2.25) is both necessary and sufficient. Finally, as by Lemma 2.1

$\rho^2\dot{\theta} = \rho^2(0)\dot{\theta}(0)$, we have that $\frac{d}{dt}(\rho^2\dot{\theta}) = 0$, which means that the left hand side of (2.23) equals zero, which means that $c_i = 0$. This completes the proof. \square

3. PROOF OF THEOREM 1.1

In Criterion 1, let $r := \rho$, $\alpha_i := \beta_i$, $\delta_i := b_i$ and $\gamma_i := c_i$. Then the conditions of Criterion 1 become exactly the conditions of Criterion 1 in [4] with the added bonus that $\gamma_i = 0$. The proof of Theorem 1.1 in [22] is therefore a proof for Theorem 1.1 as well.

4. PROOF OF THEOREM 1.2

Let again $r := \rho$, $\alpha_i := \beta_i$, $\delta_i := b_i$ and $\gamma_i := c_i$ in Criterion 1. Then the conditions of Criterion 1 become exactly the conditions of Criterion 1 in [4] with the added bonus that $\gamma_i = 0$. Theorem 1.2 now follows directly from the proofs of Theorem 1 and Theorem 2 in [4].

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